Unique RECTELLIPSE

John Jowett

In MAD-X, several beam aperture types are available (CIRCLE, RECTANGLE, ELLIPSE, ...)

In order to have a single type (homogeneous data in many optics tables, etc.), we proposed the RECTELLIPSE type, defined as the intersection of a rectangle and an ellipse, each defined by their half-axes.

This covers (almost) all the aperture types in the LHC.

RECTELLIPSE basics

Display the RECTELLIPSE description of the beam aperture in MAD.

In[3]:=
```
rectangleEllipse[{a_, b_, c_, d_}, colr_] :=
  {{Opacity[0], EdgeForm[colr], Rectangle[{-a, -b}, {a, b}]},
   {colr, Circle[{0, 0}, {c, d}]}}
```

In[4]:=
```
rectangleEllipse[{2, 1, 1.5, 1.8}, Red]
```

In[5]:=
```
aperturePlot[{a_, b_, c_, d_}] := RegionPlot[
   x^2/c^2 + y^2/d^2 < 1 && Abs[x] < a && Abs[y] < b, {x, -4, 4}, {y, -4, 4},
   Epilog → {Text[ToString[{a, b, c, d}], {0, 0}], rectangleEllipse[
      {a, b, c, d}, Directive[Thickness[0.005], Blue]]}]
```
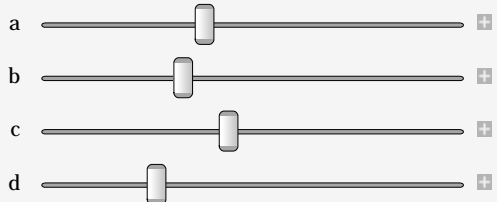
In[6]:=
```
aperturePlot[{2, 1, 1.5, 1.8}]
```
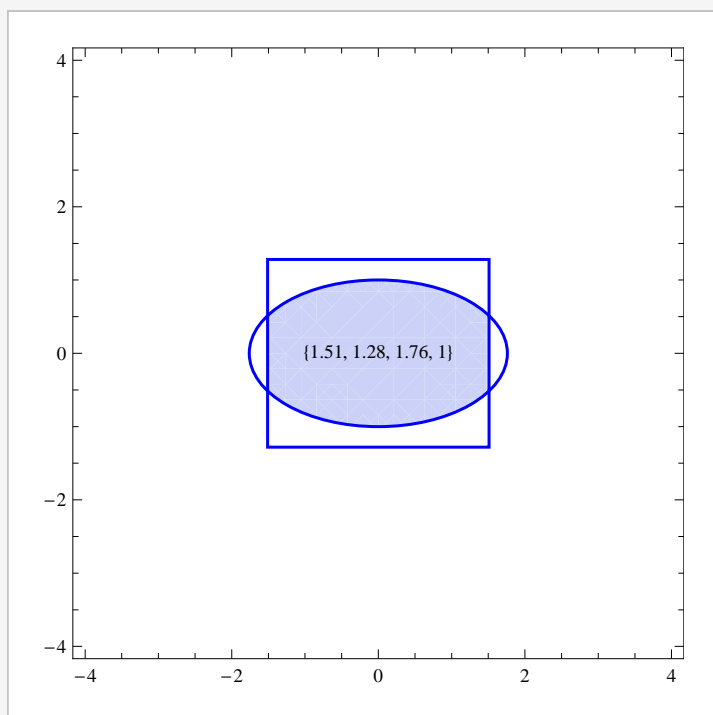
Out[6]=
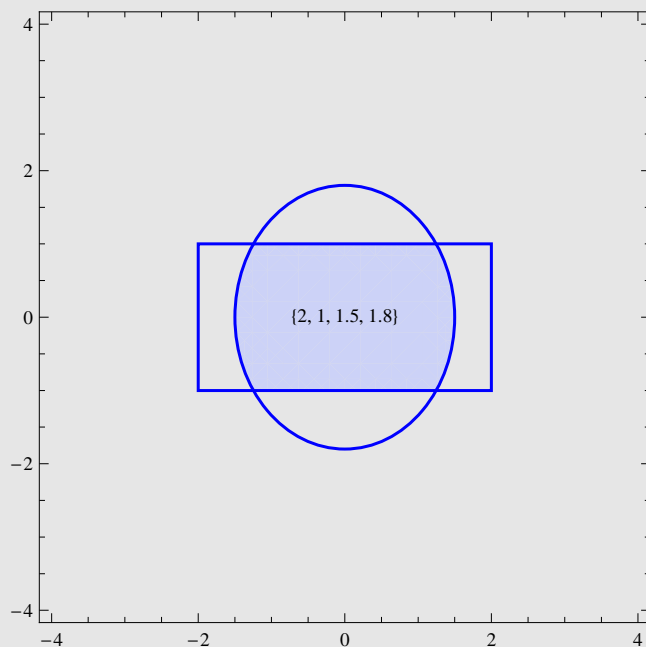
In[7]:=

```
Manipulate[
 aperturePlot[{a, b, c, d}], {{a, 1}, 0, 4},
 {{b, 1}, 0, 4}, {{c, 1}, 0, 4}, {{d, 1}, 0, 4}]
```

Out[7]=

{1.51, 1.28, 1.76, 1}

```
Manipulate[
 aperturePlot[{a, b, c, d}], {{a, 1}, 0, 4},
 {{b, 1}, 0, 4}, {{c, 1}, 0, 4}, {{d, 1}, 0, 4}]
```

How to define an *unique* RECTELLIPSE

When the rectangle is entirely inside the ellipse, we want to take the cirumscribing ellipse. The corner of the rectangle lies on this but this does not specify it uniquely. However clearly $c > a$ and $d > b$. As an additional constraint, we can take the ellipse of smallest area

$$\text{Solve}\left[\frac{a^2}{c^2} + \frac{b^2}{d^2} == 1, d\right]$$

$$\pi c d /. d \to \frac{b c}{\sqrt{-a^2 + c^2}}$$

$$D\left[\frac{b c^2 \pi}{\sqrt{-a^2 + c^2}}, c\right]$$

$$\text{Solve}\left[-\frac{b c^3 \pi}{\left(-a^2 + c^2\right)^{3/2}} + \frac{2 b c \pi}{\sqrt{-a^2 + c^2}} == 0, c\right]$$

$$d //. \left\{d \to \frac{b c}{\sqrt{-a^2 + c^2}}, c \to \sqrt{2} \ a\right\} // \text{PowerExpand}$$

◄ | ►

The other cases can be figured out and we can write down the definition of the uniqueRECTELLIPSE function:

In[8]:=
```
uniqueRECTELLIPSE[{a_, b_, c_, d_}] := Abs[Piecewise[
    {{{a, b, √2 a, √2 b}, a²/c² + b²/d² ≤ 1},
     {{c, d, c, d}, Abs[a/c] ≥ 1 && Abs[b/d] ≥ 1},
     {{a, d, c, d}, Abs[a/c] ≤ 1 && Abs[b/d] ≥ 1},
     {{c, b, c, d}, Abs[a/c] ≥ 1 && Abs[b/d] ≤ 1},
     {{a, b, c, d}, True}}
    ]]
```

Another possibility would be to take the circumscribing circle

```
uniqueRECTELLIPSE[{a_, b_, c_, d_}] := Abs[Piecewise[
    {{{a, b, Sqrt[a^2 + b^2], Sqrt[a^2 + b^2]}, a^2/c^2 + b^2/d^2 ≤ 1},
    {{c, d, c, d}, Abs[a/c] ≥ 1 && Abs[b/d] ≥ 1},
    {{a, d, c, d}, Abs[a/c] ≤ 1 && Abs[b/d] ≥ 1},
    {{c, b, c, d}, Abs[a/c] ≥ 1 && Abs[b/d] ≤ 1},
    {{a, b, c, d}, True}}
    ]]
```

Express it in traditional notation

In[9]:= **uniqueRECTELLIPSE[{a, b, c, d}] // TraditionalForm**

Out[9]//TraditionalForm=

$$\left| \begin{cases} \left\{a, b, \sqrt{2}\,a, \sqrt{2}\,b\right\} & \frac{a^2}{c^2} + \frac{b^2}{d^2} \leq 1 \\ \{c, d, c, d\} & \left|\frac{a}{c}\right| \geq 1 \bigwedge \left|\frac{b}{d}\right| \geq 1 \\ \{a, d, c, d\} & \left|\frac{a}{c}\right| \leq 1 \bigwedge \left|\frac{b}{d}\right| \geq 1 \\ \{c, b, c, d\} & \left|\frac{a}{c}\right| \geq 1 \bigwedge \left|\frac{b}{d}\right| \leq 1 \\ \{a, b, c, d\} & \text{True} \end{cases} \right|$$

Some examples:

In[10]:= **uniqueRECTELLIPSE[{1.1, 1, 1.5, 1.8}]**

Out[10]= $\left\{1.1, 1, 1.55563, \sqrt{2}\right\}$

In[11]:= **uniqueRECTELLIPSE[{-1.1, 1, 1.5, 1.8}]**

Out[11]= $\left\{1.1, 1, 1.55563, \sqrt{2}\right\}$

Using this function we can compute whether two RECTELLIPSEs are equivalent.

In[12]:= **uniqueRECTELLIPSE[{1, 1, 2, 2}] == uniqueRECTELLIPSE[{1, 1, 3, 2}]**

Out[12]= True

Another plot which shows the RECTELLIPSE components in blue and the uniqueRECTELLIPSE components in red.

In[13]:=
```
aperturePlot2[{a_, b_, c_, d_}] := RegionPlot[
    x²/c² + y²/d² < 1 && Abs[x] < a && Abs[y] < b, {x, -4, 4}, {y, -3, 3},
    PlotRange → {{-4, 4}, {-4, 4}}, Epilog → {
      Text[ToString[{a, b, c, d}], {0, 0.2}],
      Text[ToString[uniqueRECTELLIPSE[{a, b, c, d}]], {0, -0.2}],
      {rectangleEllipse[{a, b, c, d},
        Directive[Thickness[0.01], Blue]]},
      {rectangleEllipse[uniqueRECTELLIPSE[{a, b, c, d}],
        Directive[Thickness[0.005], Red]]},
      {PointSize[0.02], Blue, Point[{a, b}], Point[{c, d}]},
      {PointSize[0.014], Red,
       Point[uniqueRECTELLIPSE[{a, b, c, d}][[1 ;; 2]]],
       Point[uniqueRECTELLIPSE[{a, b, c, d}][[3 ;; 4]]]}
      }]
```
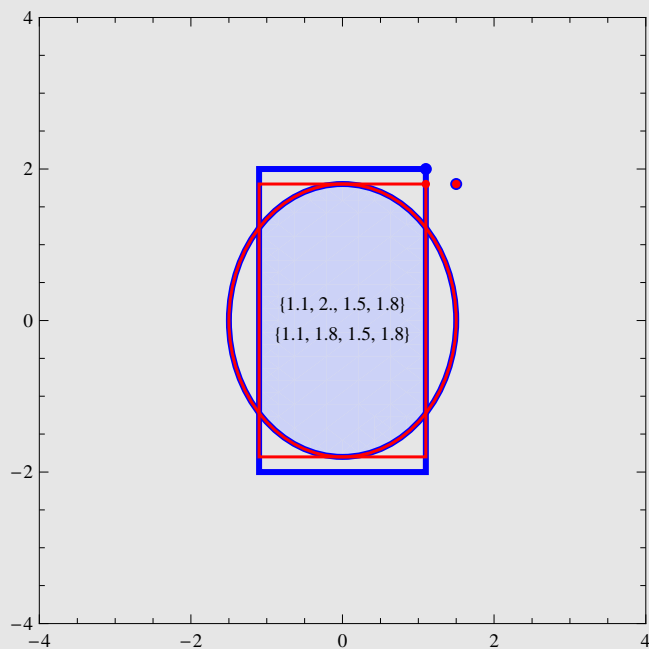
In[14]:=
```
aperturePlot2[{1.1, 2., 1.5, 1.8}]
```
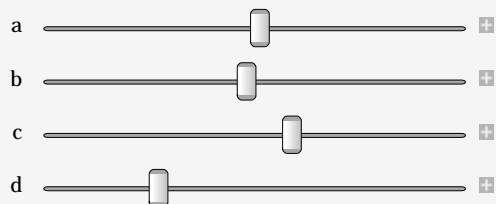
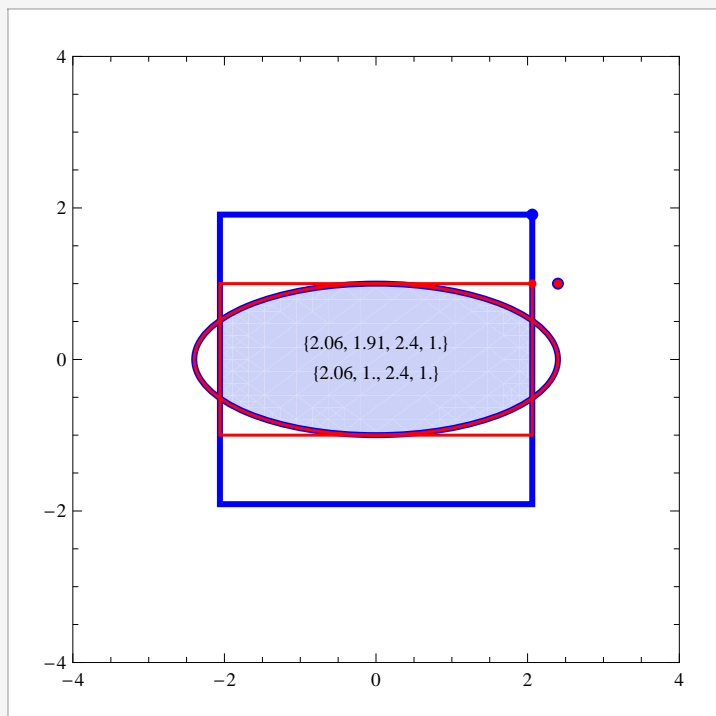Out[14]=

In[15]:=
```
Manipulate[aperturePlot2[{a, b, c, d}], {{a, 0}, 0, 4},
 {{b, 0}, 0, 4}, {{c, 1}, 0, 4}, {{d, 1}, 0, 4}]
```

Out[15]=

{2.06, 1.91, 2.4, 1.}
{2.06, 1., 2.4, 1.}

Equivalence Classes

uniqueRECTELLIPSE creates an equivalence relation on $\mathbb{R}^4$. Two points are equivalent (represent the same physical aperture) if their uniqueRECTELLIPSEs are equal.
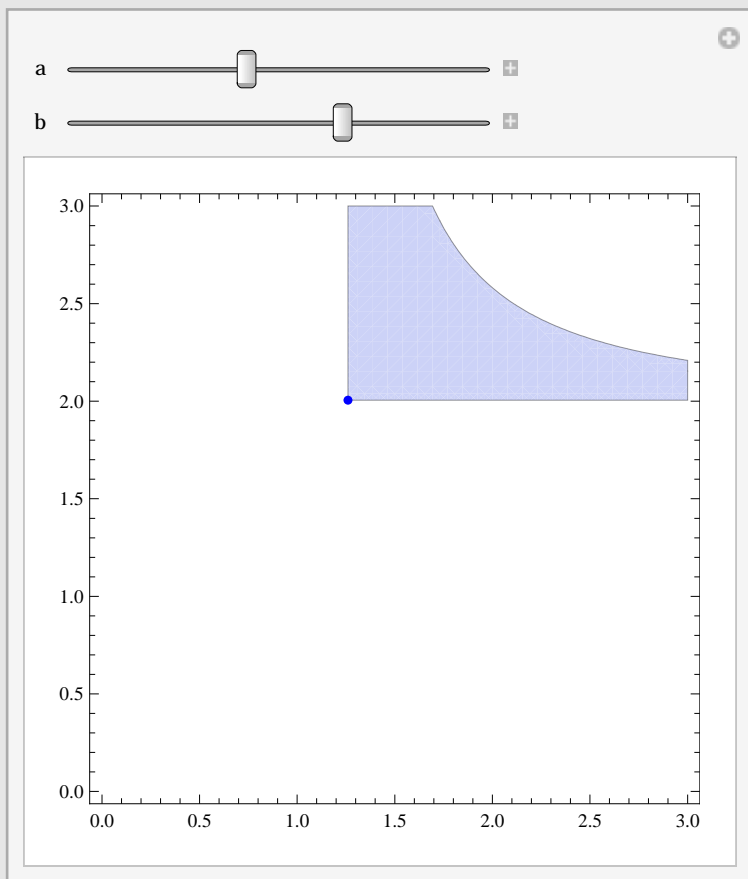
Can we give a characterisation of the equivalence classes? Or a graphical representation?

Explore this for particular values of {*a*, *b*}. The shaded region shows all the {c,d} values that provide an equivalent aperture.

In[17]:=
```
Manipulate[RegionPlot[
   {a, b, c, d} == uniqueRECTELLIPSE[{a, b, c, d}], {c, 0, 3},
   {d, 0, 3}, Epilog → {Blue, PointSize[0.015], Point[{a, b}]},
   PlotPoints → 40], {{a, 1.5}, 0, 3}, {{b, 1}, 0, 3}]
```

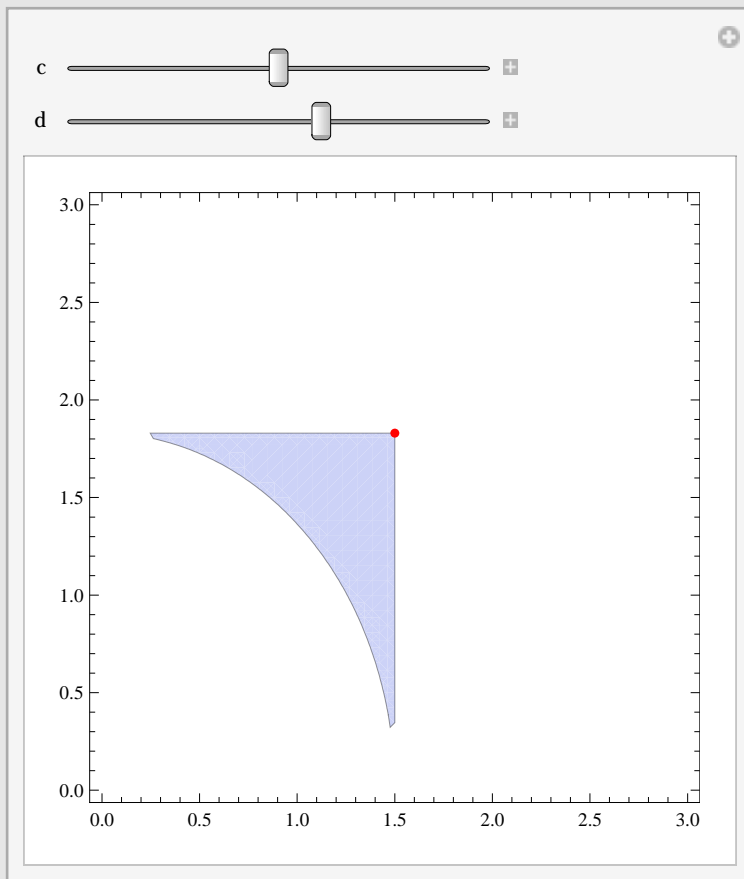Out[17]=

This time, the slider sets {c,d} and the shaded region shows the equivalent set of {a,b}

In[18]:=
```
Manipulate[RegionPlot[
   {a, b, c, d} == uniqueRECTELLIPSE[{a, b, c, d}], {a, 0, 3},
   {b, 0, 3}, Epilog → {Red, PointSize[0.015], Point[{c, d}]},
   PlotPoints → 40], {{c, 1.5}, 0, 3}, {{d, 1}, 0, 3}]
```
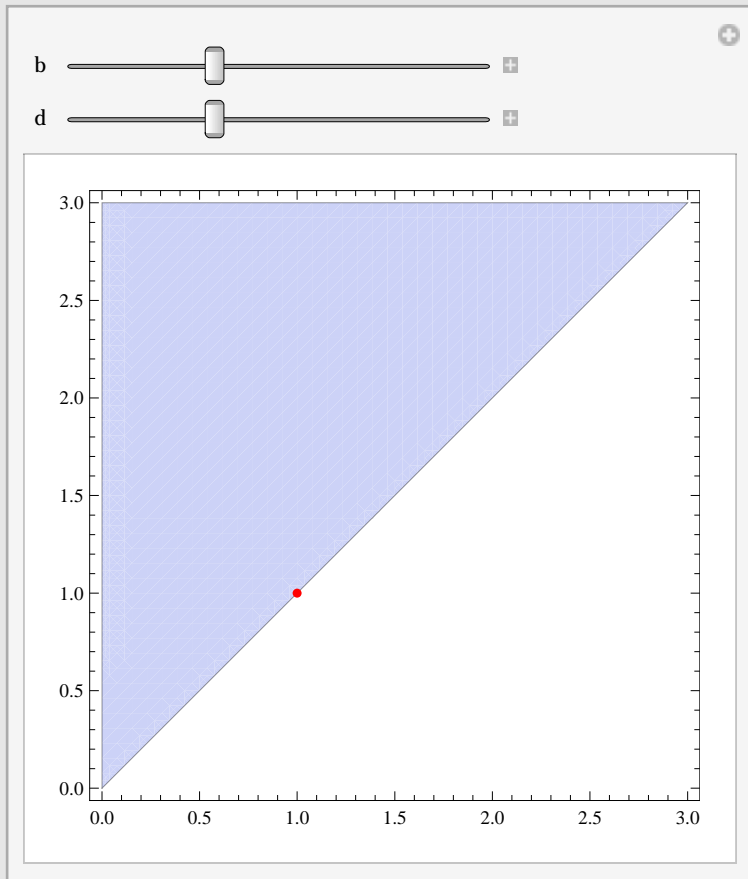
Out[18]=

In[19]:=
```
Manipulate[RegionPlot[
  {a, b, c, d} == uniqueRECTELLIPSE[{a, b, c, d}], {a, 0, 3},
  {c, 0, 3}, Epilog → {Red, PointSize[0.015], Point[{b, d}]},
  PlotPoints → 40], {{b, 1}, 0, 3}, {{d, 1}, 0, 3}]
```
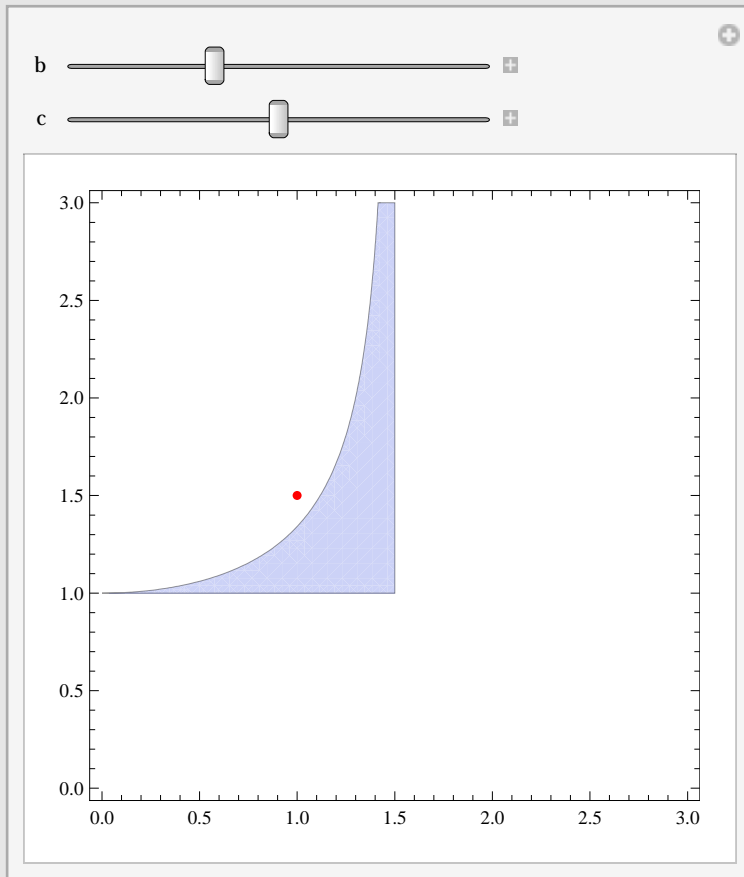
Out[19]=

```
In[20]:=   Manipulate[RegionPlot[
     {a, b, c, d} == uniqueRECTELLIPSE[{a, b, c, d}], {a, 0, 3},
     {d, 0, 3}, Epilog → {Red, PointSize[0.015], Point[{b, c}]},
     PlotPoints → 40], {{b, 1}, 0, 3}, {{c, 1}, 0, 3}]
```

Out[20]=

Summary

The definition of RECTELLIPSE with 4 parameters is conceptually simple but somewhat redundant.

The uniqueRECTELLIPSE provides a general test whether two sets of RECTELLIPSE parameters are equivalent.

It also helps to compress the volume of data required to completely specify the aperture of the LHC.