Symplectic integrator for a relativistic particle in a RF cavity

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Thanks for discussion to S. Fartoukh, E. Forest, F. Zimmermann, R. Calaga, M. Giovannozzi, L. Ficcadenti, G. Rumolo, N. Mounet, Y. Papaphilippou, R. Tomas.

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Motivation

A crab cavity model is implemented in madx and sixtrack for which particles receive an RF (uniform in x,y) kick.
Several crab cavity designs show variations of the transverse kick with the transverse position.

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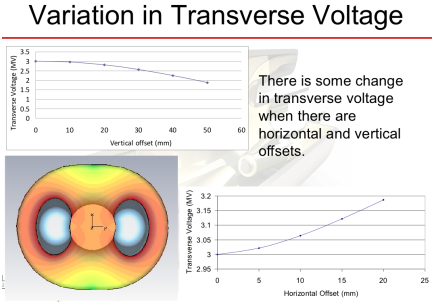

A Four Rod Compact Crab Cavity for LHC, Dr G. Burt, LHC-CC10

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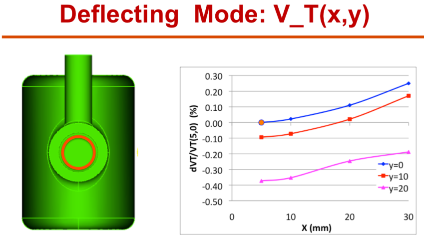

Uniformity of deflecting field
Relative V_T variation (in %)

LARP cavity developments, Z. Li, LHC-CC10

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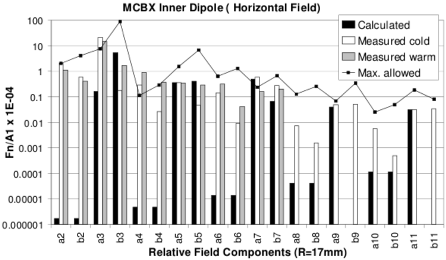

MCBX field quality

Motivation

A conclusion from the CC-10 workshop, is that crab cavity designs need a specification in terms of field quality, defined as the deviations from a pure RF dipole kick.

One possibility (among others) is to define the specifications by including a more realistic crab cavity model in sixtrack in order to observe the impact on long term simulations. Also if deviations are large, they may distort the luminous region for very low beta*.

A symplectic integrator for realistic RF cavities is proposed. It relies on the same approximations that justify the use of the wake field formalism. The simplified approach allows to describe the field of cavity by few coefficients that can be found from numerical simulations.

Assumptions

To compute the effect of an e-m fields on a relativistic particle, we assume two main approximations:

- rigid bunch approximation: the particle trajectory is not affected by the field, only momenta are affected by a kick at the end of the cavity. It is equivalent to a first order drift kick symplectic integrator.
- axial approximation: the trajectory is a straight line parallel to z, that is the motion does not depend on p_x, p_y, δ and B_s. [1].

Under this approximation the kick can be written as:

$$\Delta\vec{p}(x, y, z) = \int_{-\infty}^{\infty} \vec{F}(x, y, \beta ct - z, t)\mathrm{d}t$$

where

$$\vec{F}(x, y, s, t) = q\left[\vec{E}(x, y, s, t) + \beta c\hat{s} \times \vec{B}(x, y, s, t)\right]$$

[1] As remarked by F. Zimmermann, a para-axial approximation would keep the linear dependence on p_x, p_y, δ (e.g. solenoid). To be noted that B_s is present only in TE modes, many fringe fields are neglected for the LHC.

Panofsky-Wenzel theorem

Using Maxwell equations:

$$\nabla \cdot \vec{F} = \frac{q\rho}{\epsilon_0 \gamma^2} - \frac{q\beta}{c}\frac{\partial E_s}{\partial t} \qquad \nabla \times \vec{F} = -q\left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial s}\right)\vec{B}$$

Performing the integral we get:

$$\nabla \times \Delta\vec{p}(x,y,z) = -q\vec{B}(x,y,s=z+\beta ct,t)\Big|_{t=-\infty}^{\infty} \rightarrow 0$$

equivalent to

$$\frac{\partial}{\partial z}\Delta\vec{p}_\perp = \nabla_\perp \Delta p_s \qquad \frac{\partial \Delta p_x}{\partial y} = \frac{\partial \Delta p_y}{\partial x}$$

and

$$\nabla \cdot \Delta\vec{p}(x,y,z) = \beta\frac{\partial}{\partial z}\Delta p_s \rightarrow \frac{\partial}{\partial z}\Delta p_s$$

equivalent to

$$\frac{\partial \Delta p_x}{\partial x} + \frac{\partial \Delta p_y}{\partial y} = 0$$

Cauchy-Riemann equations

If $u(x, y)$ and $v(x, y)$ are continuous and differentiable and if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or defining $\nabla_\perp = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and $\vec{f} = (u, -v)$

$$\nabla_\perp^2 \vec{f} = 0$$

then

$$f(z = x + iy) = u(x, y) + iv(x, y) = \sum_n c_n z^n$$

Kick Hamiltonian

It can be demonstrated in absence of sources or $\beta = 1$ (Panosfky-Wenzel theorem) that:

$$\nabla \times \Delta\vec{p}(x,y,z) = 0 \qquad \nabla_\perp \cdot \Delta\vec{p}_\perp(x,y,z) = 0$$

It follows that

$$\nabla_\perp^2 \vec{p}_\perp(x,y,z) = 0$$

therefore the kick can be derived by the Hamiltonian[2]

$$H = -\Re\left[\sum_{n=0}^{\infty} W_n(z)(x+iy)^n\right] \qquad \Delta\vec{p}(x,y,z) = -\nabla H(x,y,z)$$

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[2] For magnets

$$W_n(z) = \frac{B_n + iA_n}{n r_0^{n-1}} \qquad By + iBx = \sum_{n=1}(B_n + iA_n)\left(\frac{x+iy}{r_0}\right)^{n-1}$$

RF cavity case

Since a RF cavity is excited by only one single frequency ω in phase with the synchronous particle:

- z is the delay w.r.t the synchronous particle,
- $W_n(z)$ are periodic.

If $\Delta p_\perp(x, y, z)$ is separable for $W_n(z)$ follows:

[3] $$W_n(z) = (B_n + IA_n)\cos(\omega z/c + \phi)$$

otherwise

$$W_n(z) = B_n \cos(\omega z/c + \phi_n) + iA_n \sin(\omega z/c + \psi_n)$$

In all cases, the coefficients up to the desired order for a real cavity can be computed from e-m simulations evaluating the transverse voltage for a discrete set of (x, y, z).

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[3] Already implemented in PTC by E. Forest but not avaialble in madx

Validate the approach

This approach can be validated by two distinct numerical experiments.

To validate the multiple expansion: find the coefficients up to some order using one set of field values (computed and analytical) and check that the expansion is sufficient to fit the values.

To validate the beam dynamics: simulate the particle motion through a cavity by using a detailed field map and by the multipole expansion kick. Repeat the exercise with another element class like a bending magnet or a quadrupole. Compare the level of approximation in the two cases and verify they offer consistent performance.

For improved accuracy

Probably higher order integrators could be found to keep at least to leading order of the dependence p_x, p_y, δ, B_s using the same strategy used for the fringe fields [4].
The recipe would be: include the integrated potential in the full Hamiltonian. Expand the square root of the Hamiltonian to leading orders. Use Yoshida-like integrators splitting all the integrable parts and recombine them with appropriate coefficients using the BCH theorem to minimize the deviations from the original Hamiltonian.

[4] See E. Forest, Beam Dynamics

Aknowledgment

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