effect of the crab cavity field on a particle trajectory

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Schmüser & Rossbach - basic course on accelerator optics, CAS 1992

The length of modern accelerator magnets is usually much larger than their bore radius. The end field contribution is then rather small and the magnetic field has to a good approximation only transverse components. (This is of course not the case for the large solenoids in the experimental areas which need a special treatment. The same applies for wigglers and undulators which are special magnets for generating synchrotron radiation.)

For two–dimensional fields one can apply the theory of analytic functions. From

$$div\,\mathbf{B} = 0$$

it follows that a vector potential $\mathbf{A}$ exists such that

$$\mathbf{B} = rot\,\mathbf{A} \tag{2.14}$$

Because of the transversality of the field, the vector potential has only a component A_s in the longitudinal direction s. In vacuum, for example inside the beam pipe, we have furthermore

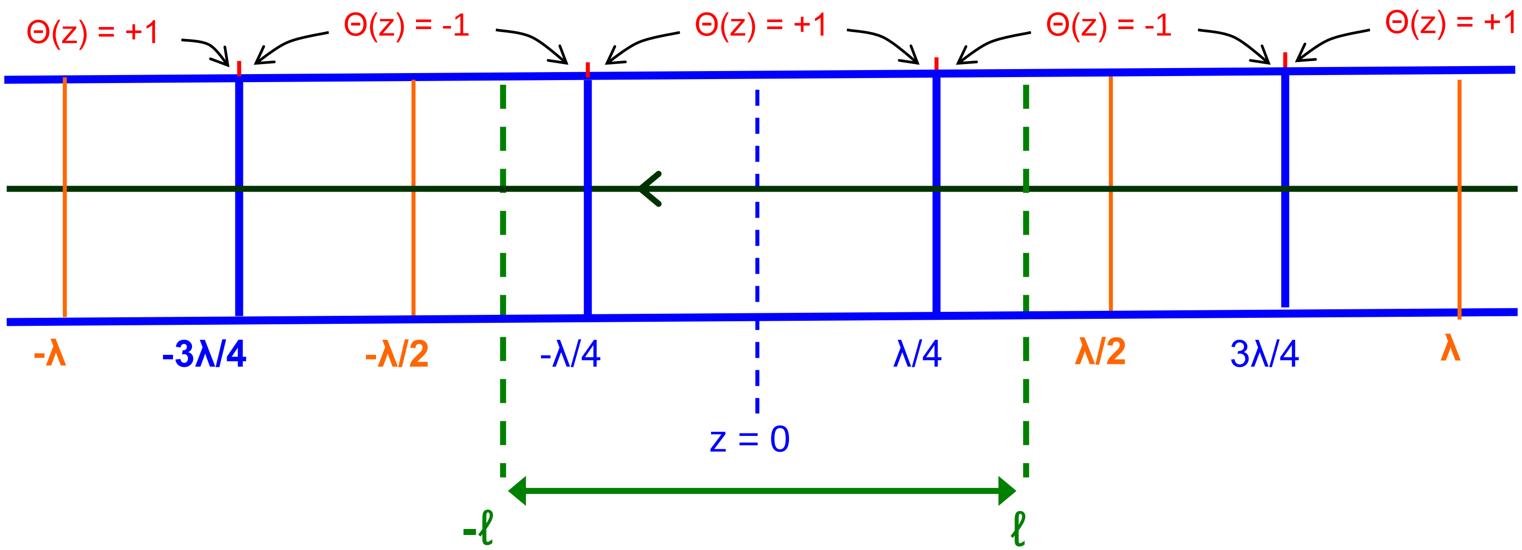
$$rot\,\mathbf{B} = 0$$

This implies that $\mathbf{B}$ can also be written as the gradient of a scalar potential V:

$$\mathbf{B} = -grad\,V \tag{2.15}$$

crab cavity fields are neither purely transverse nor static

example crab cavity



Cell Arrangement of π-mode Pill Box Cavity.
The vertical blue lines are ends of full (λ/2) cells and the vertical red lines are those of half (λ/4) cells.

Kwang-Je Kim, Trajectories through deflecting cavities, ANL-AAI-TN-2007-01, May 2007

Hybrid electromagnetic cavity mode

B. W. Montague, "Particle Separation at High Energies. II. Radiofrequency Separation", Progress in Nuclear Techniques and Instrumentation, Vol. III, North Holland, 1968.

Approximate simple expression for the electromagnetic field taking into account the non-vanishing beam-pipe radius *ρ:*

$$E_x = \mathcal{E}\frac{k}{4}\left(\rho^2 + x^2 - y^2\right)\sin ks\cos\omega t,$$

$$E_y = \mathcal{E}\frac{k}{2}\,xy\,\sin ks\cos\omega t,$$

transverse fields not uniform!

$$E_z = \mathcal{E}\,x\,\cos\,ks\cos\omega t,$$

$$cB_x = \mathcal{E}\frac{k}{2}\,xy\cos ks\sin\omega t,$$

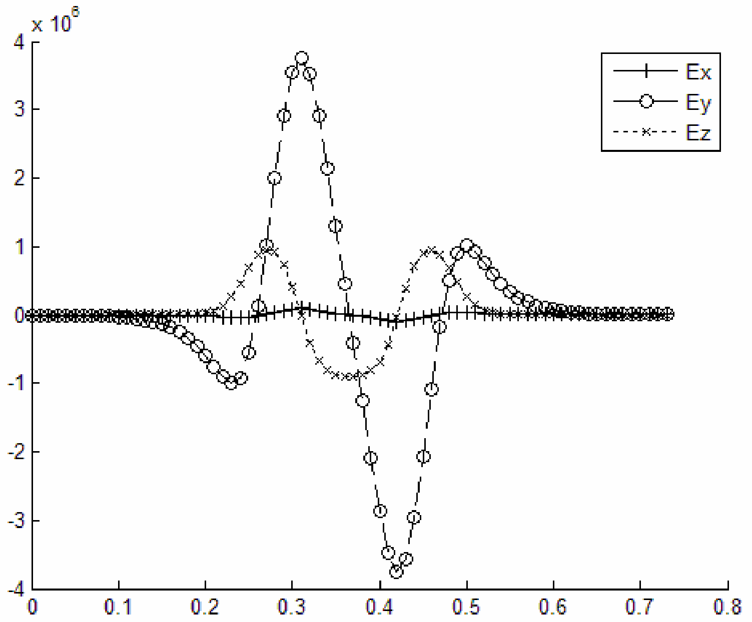
$$cB_y = -\mathcal{E}\frac{1}{k}\left(\frac{(k\rho)^2}{4} - 1 + \frac{k^2(x^2 - y^2)}{4}\right)\cos ks\sin\omega t,$$

$$cB_z = -\mathcal{E}\;y\sin ks\sin\omega t.$$

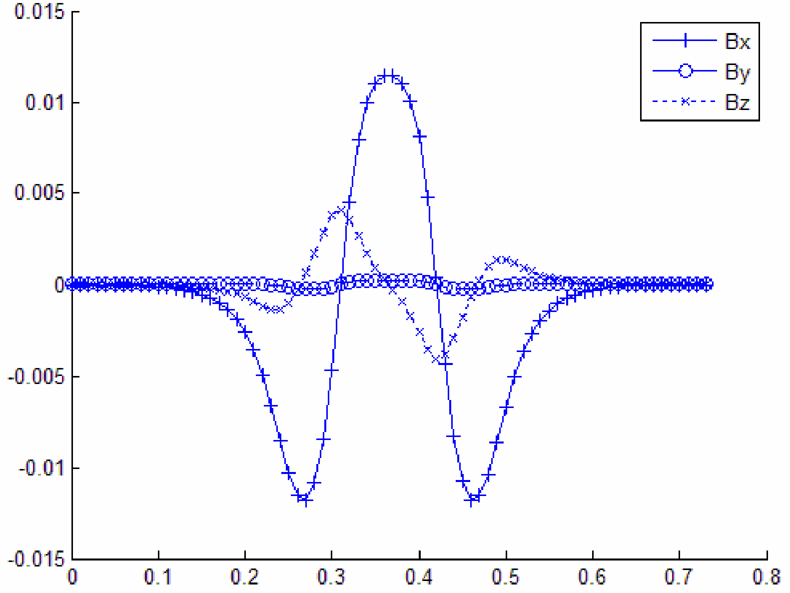
longitudinal field not zero!

Microwave studio results are very close to HEM fields up to $r=\rho$

E(V/m) vs. z (mm) at x=5mm and y=5mm



B(tesla) vs. z (mm) at x=5mm and y=5mm



Electric and magnetic field components computed from Microwave Studio for $\left(\frac{1}{2}\,1\,\frac{1}{2}\right)$ cell cavity with a beam hole.

Kwang-Je Kim, Trajectories through deflecting cavities, ANL-AAI-TN-2007-01, May 2007

neglecting quadratic terms,
trajectory equations for a HEM cavity are:

$$\frac{d^2x}{ds^2} = \frac{G}{k}\left\{\xi \sin(2ks-\theta) - \frac{1}{2}\left[\sin(2ks-\theta)-\sin\theta\right]\right\}$$

$$\frac{d\delta}{ds} = \frac{G}{2k}\{kx[\cos(2ks-\theta)+\cos\theta]+x'(1-\xi)[\sin(2ks-\theta)-\sin\theta]\}$$

where $k=\omega/c=2\pi/\lambda,\qquad \theta=k\,z,$

$$G=\frac{\mathcal{E}}{E_e},\qquad \xi=\left(\frac{k\rho}{2}\right)^2,\qquad \phi=k\,L/2$$

solution for *x*:

$$x_2' = x_1' + \frac{G}{k} z \left(\phi - \left(\xi - \frac{1}{2} \right) \sin 2\phi \right)$$

$$x_2 = x_1 + L x_1' + \frac{G}{2k^3} \left(\xi - \frac{1}{2} \right) \left(2\phi \cos 2\phi - \sin 2\phi \right),$$

$$+ \frac{G}{k^2} z \left[\phi - \left(\xi - \frac{1}{2} \right) \sin 2\phi \right] \phi.$$

constant offset in *x* which does not appear in case of a multipole description

Kwang-Je Kim, Trajectories through deflecting cavities, ANL-AAI-TN-2007-01, May 2007

generalized multipoles for a **general static *B* field** with dipole geometry:

Y. Papaphilippou, J. Wei, R. Talman, EPAC2000 & PRE 67, 046502, 2003

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+2-2l}^{[2l]}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m}}{(2n)!(2m)!} \binom{m}{l} b_{2n+2m-2l}^{[2l]}$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m-2l}^{[2l+1]}$$

Taking the field expansion up to leading order, we get:

$$\begin{aligned} B_x &= b_2 xy + O(4) \\ B_y &= b_0 - \frac{1}{2} b_0^{[2]} y^2 + \frac{1}{2} b_2 (x^2 - y^2) + O(4) \qquad (2) \\ B_z &= y\, b_0^{[1]} + O(3) \end{aligned}$$

terms which differ from regular multipoles

where b_2 represents a sextupole field component allowed by the symmetry of the "dipole" magnet (for an ideally designed magnet $b_2 = 0$) and $O(3)$ and $O(4)$ contain all the allowed terms of higher orders.

generalized "multipole" expression for crab cavities? (*E+B* field, harmonic time dependence)

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$$

Lorentz force

$$\vec{F} = e\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) + e\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)$$

$$-\Delta\vec{\boldsymbol{A}} - \frac{\boldsymbol{\omega}^2}{\boldsymbol{c}^2}\vec{\boldsymbol{A}} = \boldsymbol{0}$$

vector Helmholtz equation

$$\vec{\nabla} \cdot \vec{A} - i\frac{\omega}{c^2}\Phi = 0$$

Lorentz condition (gauge)

$$-\vec{\nabla}\left(\vec{\nabla} \cdot \vec{\mathrm{A}}\right) = -\Delta\vec{\mathrm{A}} - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)$$

vector identity

$$\vec{F} = e\left(-\vec{\nabla}\frac{c^2}{i\omega}\left(\vec{\nabla} \cdot \vec{\mathrm{A}}\right) + \mathrm{i}\omega\vec{\mathrm{A}}\right) + e\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)$$

$$\vec{F} = e\left(i\frac{c^2}{\omega}\vec{\nabla} + \vec{v}\right) \times \left(\vec{\nabla} \times \vec{A}\right) \quad \rightarrow \quad \vec{\boldsymbol{F}} = \boldsymbol{e}\left(\boldsymbol{i}\frac{\boldsymbol{c}^2}{\boldsymbol{\omega}}\vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{v}}\right) \times \vec{\boldsymbol{B}}$$

solution of the scalar Helmholtz equation in (r, θ, z)

we need to solve this equation for all 3 components of vector potential with appropriate symmetry & boundary conditions

$$\nabla^2 F + k^2 F = 0$$

$$F(r, \theta, z) = R(r)\,\Theta(\theta)\,Z(z),$$

look into this with Yannis?

$$\Theta(\theta) = C_m \cos(m\,\theta) + D_m \sin(m\,\theta).$$

$$R(r) = A_{mn}\, J_m\left(r\sqrt{n^2+k^2}\right) + B_{mn}\, Y_m\left(r\sqrt{n^2+k^2}\right).$$

general solution:

$$F(r, \theta, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[A_{mn}\, J_m\left(r\sqrt{k^2+n^2}\right) + B_{mn}\, Y_m\left(r\sqrt{k^2+n^2}\right)\right]$$

$$\times\left[C_m \cos(m\,\theta) + D_m \sin(m\,\theta)\right]\left(E_n\, e^{-n z} + F_n\, e^{n z}\right).$$

from *Wolfram MathWorld*

preliminary conclusions

transverse multipole expansion misses
terms which are related to z-dependence and
terms which are related to time dependence

to rely on such expressions one must make sure
that the neglected terms are small

even the constant position offset may be significant;
so how about higher order terms?

generalized multipoles are known for a nonuniform
static magnetic field; even more generalized multipoles
might be derived for the case with time dependence