Thin lens slicing

Thin lens version of magnet lattice : for tracking and error assignment

The makethin module in MAD-X allows for automatic slicing

```
MAKETHIN,SEQUENCE=sequence name, STYLE=slicing style;
where STYLE
SIMPLE : this is a simplified slicing algorithm which produces any number of equal strength
slices at equidistant positions with the kick in the middle of each slice.
TEAPOT (default): this is the standard slicing. It has a maximum of four slices for any one
object.
```

Much used with MAD-X and SIXTRACK for the LHC
TEAPOT is much better than SIMPLE, described in recent IPAC'12 paper

Tracking LHC Models with Thick Lens Quadrupoles: Results and Comparisons with the Standard Thin Lens Tracking, tuppc079

Here : extending **TEAPOT** slicing to any $n > 1$, by minimizing the 3rd order focusing term in the transfer matrix

Makethin updated and ATS Note with details written, to be released soon

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Massimo Giovannozzi, Thys Risselada, John Jowett, Werner Herr, Laurent Deniau, Riccardo De Maria

Matrices, thick/thin quadrupole and drift

For the discussion sufficient to work in 2-dimensions x, x' (or equivalently y,y')

$$\mathbf{M}_{\text{thick}}(K, L) = \begin{pmatrix} \cos KL & \frac{\sin KL}{K} \\ -K \sin KL & \cos KL \end{pmatrix}$$

$$\mathbf{M}_{\text{drift}}(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M}_{\text{thin}}(KL) = \begin{pmatrix} 1 & 0 \\ -K^2 L & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

2,1 matrix element, focusing term

where

K^2 quadrupole strength and L its length, $f = 1 / K^2 L$ the focal length

d is the drift length

Simple equidistant slicing

$$\mathbf{M}_{\text{simple}}(K, L, n) = \left(\mathbf{M}_{\text{drift}}(\frac{L}{2n})\ \mathbf{M}_{\text{thin}}(\frac{KL}{n})\ \mathbf{M}_{\text{drift}}(\frac{L}{2n})\right)^n$$

Taylor expansions of the 2,1 matrix elements to 5th order in L

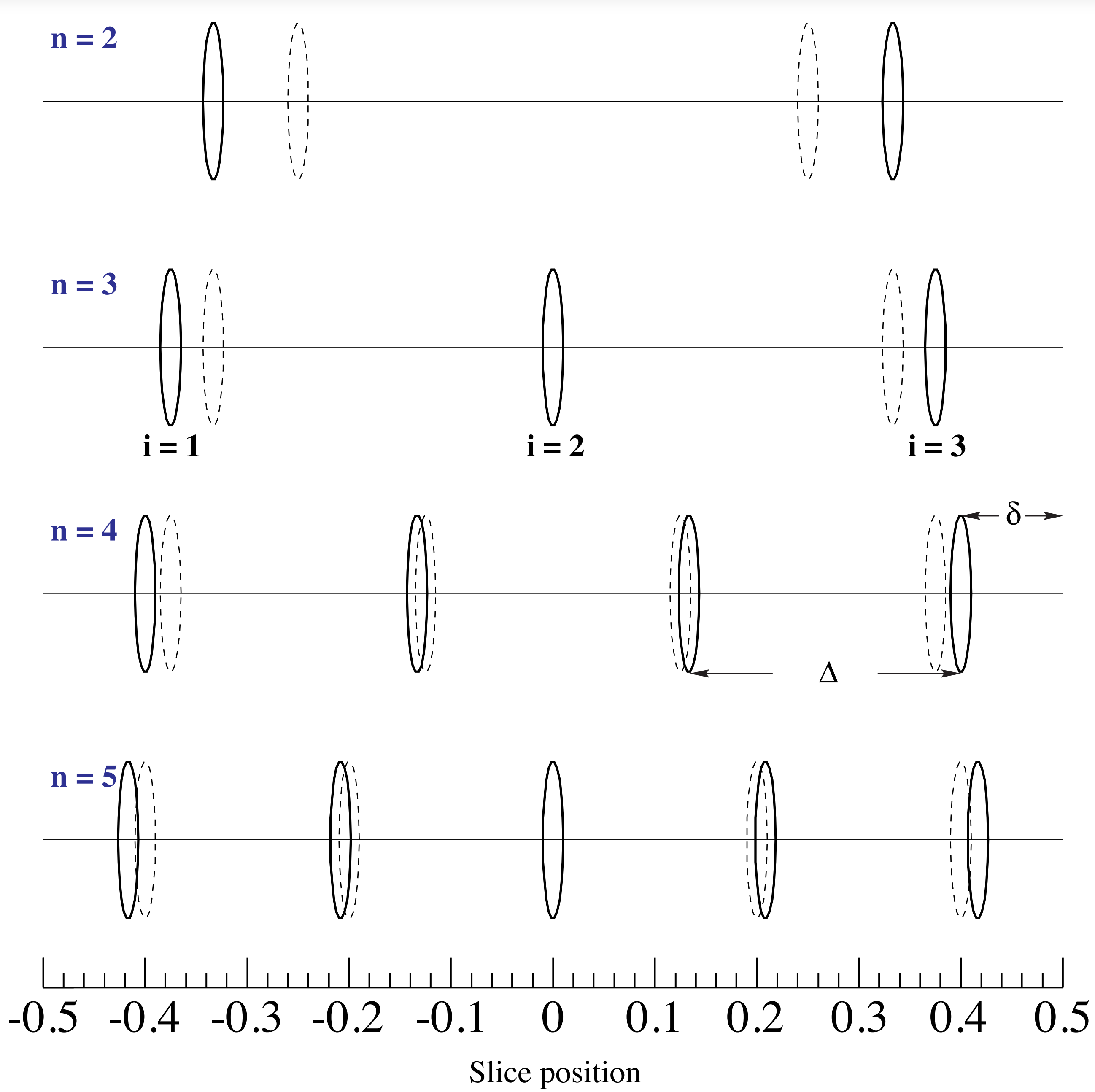
$$\mathbf{M}_{\text{thick}}(K, L)_{2,1} = -K^2 L \left(1 - \frac{K^2 L^2}{6} + \frac{K^4 L^4}{120}\right)$$

$$\mathbf{M}_{\text{simple}}(K, L, n)_{2,1} = -K^2 L \left(1 - \frac{K^2 L^2}{6}\left(1 - \frac{1}{n^2}\right) + \frac{K^4 L^4}{120}\left(1 - \frac{5}{n^2} + \frac{4}{n^4}\right)\right)$$

differ in 3rd order in L

This can be fixed by moving the thin slices a bit closer to the edge ---> TEAPOT slicing

SIMPLE and TEAPOT slicing

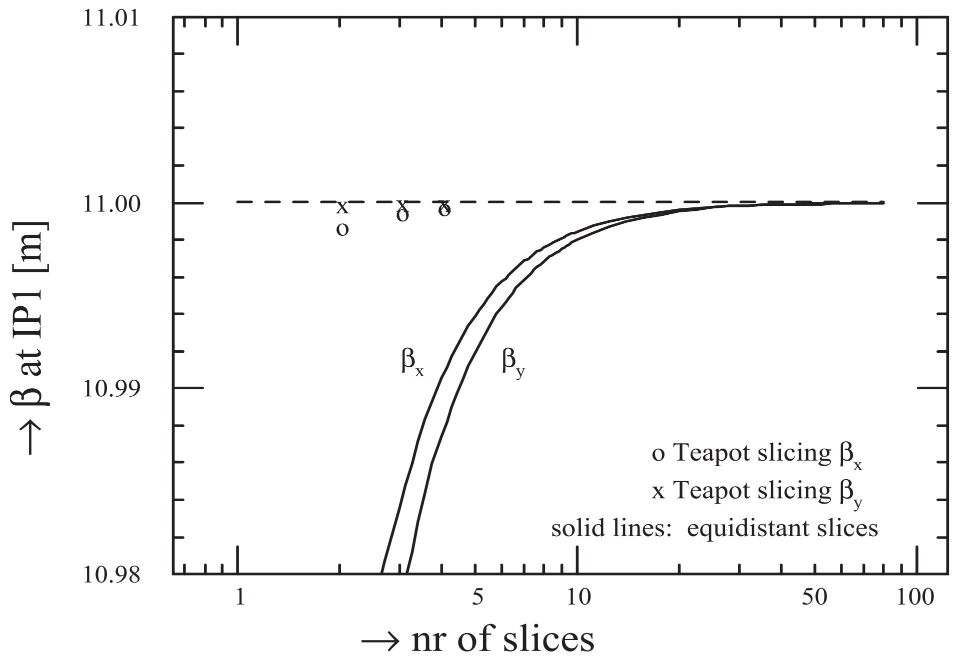


n slices

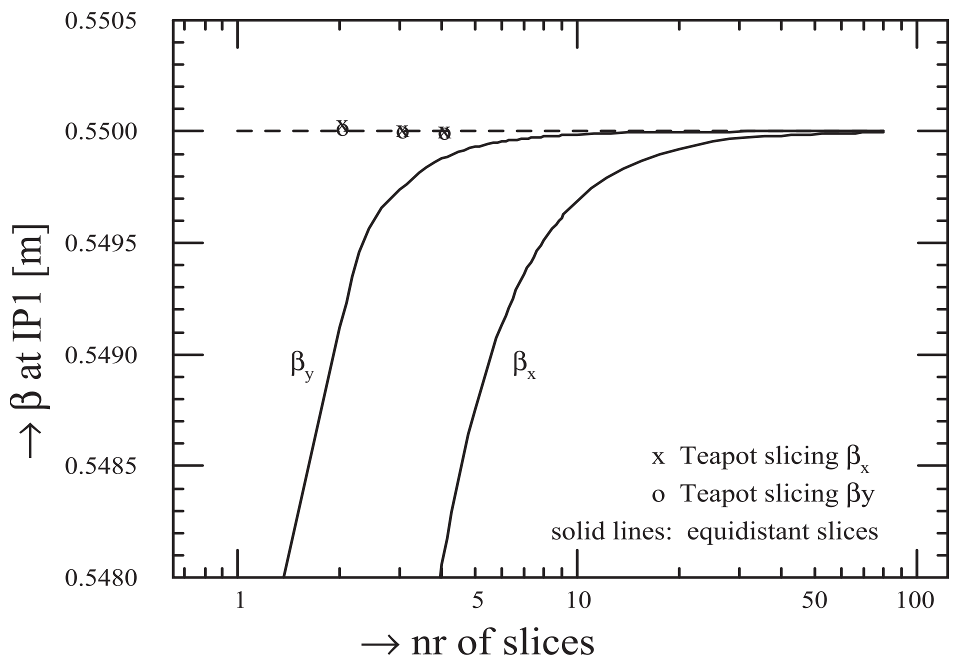
i - th slice

one single new parameter δ, distance to the edge in units of L $2\delta + (n - 1)\,\Delta = 1$

TEAPOT much better than SIMPLE



LHC injection



LHC collision optics

TEAPOT slicing

$$\mathbf{M}_{\text{teapot}}(K,L,n,\delta) = \mathbf{M}_{\text{drift}}(L\,\delta)\ \left(\mathbf{M}_{\text{thin}}(\frac{KL}{n})\ \mathbf{M}_{\text{drift}}(L\,\Delta)\right)^{(n-1)} \mathbf{M}_{\text{thin}}(\frac{KL}{n})\ \ \mathbf{M}_{\text{drift}}(L\,\delta)$$

Taylor expansions to 5th order on L

$$\mathbf{M}_{\text{teapot}}(K,L,n,\delta)_{2,1} = -K^2L\left[1-\frac{K^2L^2}{6}\underbrace{\left(1+\frac{1}{n}\right)(1-2\delta)}_{=1}+\frac{K^4L^4}{120}\frac{(n^2-4)(n+1)}{n^2(n-1)}(1-2\delta)^2\right]$$

= 1 for $\delta = \frac{1}{2}\frac{1}{1+n}$

so that

$$\mathbf{M}_{\text{teapot}}(K,L,n)_{2,1} = -K^2L\left[1-\frac{K^2L^2}{6}+\frac{K^4L^4}{120}\frac{n^2-4}{n^2-1}\right]$$

which agrees with the thick matrix element to L^3

$$\mathbf{M}_{\text{thick}}(K,L)_{2,1} \quad = \quad -K^2L\left(1-\frac{K^2L^2}{6}+\frac{K^4L^4}{120}\right)$$

TEAPOT slicing

n	δ	Δ teapot	Δ simple
2	1/6	$n/3 = 0.6666$	$1/n = 0.5$
3	1/8	$n/8 = 0.375$	$1/n = 0.33333$
4	1/10	$n/15 = 0.2666$	$1/n = 0.25$
n	$1/(2(1+n))$	$n/(n^2-1)$	$1/n$

old code in makethin, limited to 4

```
static double
teapot_at_shift(int slices,int slice_no)
{
  double at = 0;
  switch (slices)
  {
    case 1:
      at = 0.;
      break;
    case 2:
      if (slice_no == 1) at = -1./3.;
      if (slice_no == 2) at = +1./3.;
      break;
    case 3:
      if (slice_no == 1) at = -3./8.;
      if (slice_no == 2) at = 0.;
      if (slice_no == 3) at = +3./8.;
      break;
    case 4:
      if (slice_no == 1) at = -2./5.;
      if (slice_no == 2) at = -2./15.;
      if (slice_no == 3) at = +2./15.;
      if (slice_no == 4) at = +2./5.;
      break;
  }
  /* return the simple style if slices > 4 */
  if (slices > 4) at = simple_at_shift(slices,slice_no);
  return at;
}
```

new code, OK for any n

```
inline double teapot_at_shift(int n,int i)

{return ( (n>1) ? (0.5*n*(1-2*i+n)/(1-n*n)) :0 );}
```

result exactly the same as before for n = 2, 3, 4

References

TEAPOT

L. Schachinger and R. Talman, "TEAPOT: A thin element accelerator program for optics and tracking", *Part.Accel.* 22 (1987) 35, SSC-052.

R. Talman, "Representation of Thick Quadrupoles by Thin Lenses", SSC-N-033, August 1985.

LHC, MAD-X

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Symplectic integration

E. Forest and R. Ruth, "Fourth order symplectic integration", *Physica* D43 (1990) 105–117.

H. Yoshida, "Construction of higher order symplectic integrators", *Phys.Lett.* A150 (1990) 262–268.

A. Chao, "Lecture notes on topics in accelerator physics", SLAC-PUB-9574.

Backup

Matrices to 5-order

$$\mathbf{M}_{\text{thick}}(K,L) = \begin{pmatrix} 1-\frac{K^2L^2}{2}+\frac{K^4L^4}{24} & L\left(1-\frac{K^2L^2}{6}+\frac{K^4L^4}{120}\right) \\ -K^2L\left(1-\frac{K^2L^2}{6}+\frac{K^4L^4}{120}\right) & 1-\frac{K^2L^2}{2}+\frac{K^4L^4}{24} \end{pmatrix}$$

$$\mathbf{M}_{\text{teapot}}(K,L,n) = \begin{pmatrix} 1-\frac{K^2L^2}{2}+\frac{K^4L^4}{24}\frac{n^2-2}{n^2-1} & L\left(1-\frac{K^2L^2}{6}\frac{2n^2-3}{2n^2-2}+\frac{K^4L^4}{120}\frac{n^4-4n^2+5}{n^4-2n^2+1}\right) \\ -K^2L\left(1-\frac{K^2L^2}{6}+\frac{K^4L^4}{120}\frac{n^2-4}{n^2-1}\right) & 1-\frac{K^2L^2}{2}+\frac{K^4L^4}{24}\frac{n^2-2}{n^2-1} \end{pmatrix}$$

$$\mathbf{M}_{\text{simple}}(K,L,n) =$$

$$\begin{pmatrix} 1-\frac{K^2L^2}{2}+\frac{K^4L^4}{24}\left(1-\frac{1}{n^2}\right) & L\left(1-\frac{K^2L^2}{6}\left(1+\frac{1}{2n^2}\right)+\frac{K^4L^4}{120}\left(1-\frac{1}{n^4}\right)\right) \\ -K^2L\left(1-\frac{K^2L^2}{6}\left(1-\frac{1}{n^2}\right)+\frac{K^4L^4}{120}\left(1-\frac{5}{n^2}+\frac{4}{n^4}\right)\right) & 1-\frac{K^2L^2}{2}+\frac{K^4L^4}{24}\left(1-\frac{1}{n^2}\right) \end{pmatrix}$$

Matrices to 5-order

Rewrite this in terms of focal length $f = \frac{1}{K^2 L}$, and the ratio of length to focal length $r = L/f$,

$$\mathbf{M}_{\text{thick}}(K, L) = \begin{pmatrix} 1 - \frac{r}{2} + \frac{r^2}{24} & L\left(1 - \frac{r}{6} + \frac{r^2}{120}\right) \\ -\frac{1}{f}\left(1 - \frac{r}{6} + \frac{r^2}{120}\right) & 1 - \frac{r}{2} + \frac{r^2}{24} \end{pmatrix}$$

$$\mathbf{M}_{\text{teapot}}(K, L, n) = \begin{pmatrix} 1 - \frac{r}{2} + \frac{r^2}{24}\frac{n^2-2}{n^2-1} & L\left(1 - \frac{r}{6}\frac{2n^2-3}{2n^2-2} + \frac{r^2}{120}\frac{n^4-4n^2+5}{n^4-2n^2+1}\right) \\ -\frac{1}{f}\left(1 - \frac{r}{6} + \frac{r^2}{120}\frac{n^2-4}{n^2-1}\right) & 1 - \frac{r}{2} + \frac{r^2}{24}\frac{n^2-2}{n^2-1} \end{pmatrix}$$

$$\mathbf{M}_{\text{simple}}(K, L, n) =$$

$$\begin{pmatrix} 1 - \frac{r}{2} + \frac{r^2}{24}\left(1 - \frac{1}{n^2}\right) & L\left(1 - \frac{r}{6}\left(1 + \frac{1}{2n^2}\right) + \frac{r^2}{120}\left(1 - \frac{1}{n^4}\right)\right) \\ -\frac{1}{f}\left(1 - \frac{r}{6}\left(1 - \frac{1}{n^2}\right) + \frac{r^2}{120}\left(1 - \frac{5}{n^2} + \frac{4}{n^4}\right)\right) & 1 - \frac{r}{2} + \frac{r^2}{24}\left(1 - \frac{1}{n^2}\right) \end{pmatrix}$$

Typical numbers, LHC

Arc quadrupoles $L = 3.1$ m $K = \sqrt{0.0088}$ /m $KL = 0.291$, $f = 36.65$ m, $r = 0.84568$

$$\text{Mthick} = \begin{pmatrix} 0.958013 & 3.05649 \\ -0.0268971 & 0.958013 \end{pmatrix}$$

$$\text{Msimple} = \begin{pmatrix} 0.957939 & 3.05102 \\ -0.0269916 & 0.957939 \end{pmatrix}$$

$$\text{Mteapot} = \begin{pmatrix} 0.957915 & 3.06369 \\ -0.0268955 & 0.957915 \end{pmatrix}$$

2 slices

relative error

$$\begin{pmatrix} 0.0000768869 & 0.00179057 \\ -0.0035135 & 0.0000768869 \end{pmatrix}$$

$$\begin{pmatrix} 0.000102808 & -0.00235581 \\ 0.0000603247 & 0.000102808 \end{pmatrix}$$

Teapot nearly 60× better in M21
For the other matrix elements the precision is similar to simple slicing